THU-96/14
hep-th/9603191

N=2 SYMPLECTIC REPARAMETRIZATIONS IN A CHIRAL BACKGROUND[†]

B. de Wit
Institute for Theoretical Physics, Utrecht University
Princetonplein 5, 3508 TA Utrecht, The Netherlands

ABSTRACT

We study the symplectic reparametrizations that are possible for theories of $N = 2$ supersymmetric vector multiplets in the presence of a chiral background and discuss some of their consequences. One of them concerns an anomaly arising from a conflict between symplectic covariance and holomorphy.

[†] Based on a talk given at the Workshop on Recent Developments in Theoretical Physics, "STU-Dualities and Nonperturbative Phenomena in Superstrings and Supergravity", Cern Theory Division, 29 November - 1 December, 1995; to appear in Fortschritte der Physik.

March 1996

N=2 SYMPLECTIC REPARAMETRIZATIONS IN A CHIRAL BACKGROUND

B. DE WIT
Institute for Theoretical Physics, Utrecht University
Princetonplein 5, 3508 TA Utrecht, The Netherlands

ABSTRACT

We study the symplectic reparametrizations that are possible for theories of $N = 2$ supersymmetric vector multiplets in the presence of a chiral background and discuss some of their consequences. One of them concerns an anomaly arising from a conflict between symplectic covariance and holomorphy.

1. Introduction

Theories of abelian $N = 2$ vector multiplets transform systematically under duality transformations: transformations acting on the (abelian) field strengths which rotate the combined field equations and Bianchi identities by means of a real symplectic matrix. This was first exploited for pure $N = 2$ supergravity [1]. For generic $N = 2$ vector supermultiplets it was discovered [2] that these transformations rotate the scalar fields X^I and the derivatives F_I of the holomorphic function $F(X)$ that encodes the Lagrangian, by means of the same $Sp(2n+2;\mathbf{R})$ transformation, where n denotes the number of vector multiplets[1]. Initially the emphasis was on invariances of the equations of motion. The fact that the scalars in supergravity often parametrize an homogeneous space whose transitive isometries are realized through duality transformations, enables one to conveniently controll the nonpolynomial dependence on the scalar fields. Later it was realized that these transformations can also be used to reparametrize the theory in terms of a different function $\tilde{F}(\tilde{X})$ [3]. For the subgroup of the symplectic group corresponding to an invariance of the equations of motion, the function F will remain the same.

The same symplectic reparametrizations emerged in the context of type-II string compactifications on Calabi-Yau manifolds, where (X^I, F_J) can be associated with the periods of the $(3,0)$ form of the Calabi-Yau three-fold. These periods transform under symplectic rotations induced by changes in the corresponding homology basis [4, 5]. The scalar sector of the vector multiplets, which in this application corresponds to (part of) the moduli space of the Calabi-Yau manifolds, are therefore subject to the same symplectic transformations.

More recently symplectic reparametrizations were exploited by Seiberg and Witten [6], and later by others [7], in obtaining exact solutions of low-energy effective

[1]Not counting the graviphoton of $N = 2$ supergravity. In the rigid case the symplectic matrix is only $2n$-dimensional.

actions for $N = 2$ supersymmetric Yang-Mills theory. Singularities in these effective actions signal their breakdown due to the emergence of massless states corresponding to monopoles and dyons. Although these states are the result of nonperturbative dynamics, they are nevertheless accessible because at these points one conveniently converts to an alternative dual formulation, in which local field theory is again applicable. The same ideas have been used in the description of the effective action of heterotic $N = 2$ compactifications [8-10].

Both in the context of Calabi-Yau manifolds and in the nonperturbative sector of supersymmetric Yang-Mills theories, the symplectic transformations are usually restricted to a discrete subgroup. This can be understood from the fact that they are related to changes of the homology basis, the periodicity of the generalized θ-angles and/or the invariant rotations of the lattice of electric-magnetic charges.

In this talk we discuss a number of features related to the symplectic reparametrizations in the presence of a chiral background. As some of this material has already been covered elsewhere [11], we will mainly summarize some of the results and clarify specific points.

2. Symplectic reparametrizations

The actions we use are based on $N = 2$ chiral superspace integrals,

$$S \propto \mathrm{Im}\Big(\int \mathrm{d}^4x\,\mathrm{d}^4\theta\; F(W^I)\Big)\,, \tag{1}$$

where F is is an arbitrary function of reduced chiral multiplets $W^I(x,\theta)$. Such multiplets carry the gauge-covariant degrees of freedom of a vector multiplet, consisting of a complex scalar X^I, a spinor doublet Ω^{iI}, an anti-selfdual field-strength $F^{-I}_{\mu\nu}$ and a triplet of auxiliary fields Y^I_{ij}. This Lagrangian may coincide with the effective Lagrangian associated with some supersymmetric Yang-Mills theory, but for our purposes its origin is not relevant. To enable coupling to supergravity the holomorphic function must be homogeneous of second degree.

The Lagrangian contains spin-1 kinetic terms proportional to

$$\mathcal{L} \propto i\Big(\mathcal{N}_{IJ}\,F^{+I}_{\mu\nu}F^{+\mu\nu J} \;-\; \bar{\mathcal{N}}_{IJ}\,F^{-I}_{\mu\nu}F^{-\mu\nu J}\Big)\,, \tag{2}$$

where $F^{\pm I}_{\mu\nu}$ are the (anti-)selfdual field strengths and $\mathcal{N}$ is proportional to the second derivative of the function $\bar F(\bar X)$. In addition there are moment couplings (to the fermions, or to certain background fields, to be discussed later), so that the field strengths $F^{\pm I}_{\mu\nu}$ couple linearly to tensors $\mathcal{O}^{\pm\mu\nu}_I$, whose form is left unspecified at the moment. Define

$$G^{+}_{\mu\nu I} = \mathcal{N}_{IJ}F^{+J}_{\mu\nu} + \mathcal{O}^{+}_{\mu\nu I}\,, \tag{3}$$

and the corresponding anti-selfdual tensor that follows from complex conjugation, so that the field equations read $\partial^\mu(G^{+}_{\mu\nu I} - G^{-}_{\mu\nu I}) = 0$. The Bianchi identities and equa-

tions of motion for the Abelian gauge fields are invariant under the transformation

$$\begin{pmatrix} F^{\pm I}_{\mu\nu} \\ G^{\pm}_{\mu\nu I} \end{pmatrix} \longrightarrow \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F^{\pm I}_{\mu\nu} \\ G^{\pm}_{\mu\nu I} \end{pmatrix} , \tag{4}$$

where U^I_J, $V_I{}^J$, W_{IJ} and Z^{IJ} are constant real $(n+1)\times(n+1)$ submatrices. From (3) and (4) one derives that $\mathcal{N}$ must transform as

$$\tilde{\mathcal{N}}_{IJ} = (V_I{}^K\mathcal{N}_{KL} + W_{IL})\,[(U+Z\mathcal{N})^{-1}]^L{}_J\,. \tag{5}$$

To ensure that $\mathcal{N}$ remains a symmetric tensor, at least in the generic case, the transformation (4) must be an element of $Sp(2n+2,\mathbf{R})$ (we disregard a uniform scale transformation). Owing to this restriction, the signature of the imaginary part of $\mathcal{N}$ is invariant. Furthermore the tensor $\mathcal{O}$ must change according to

$$\tilde{\mathcal{O}}^+_{\mu\nu I} = \mathcal{O}^+_{\mu\nu J}\,[(U+Z\mathcal{N})^{-1}]^J{}_I\,. \tag{6}$$

The required change of $\mathcal{N}$ is induced by a change of the scalar fields, implied by

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix} . \tag{7}$$

The two transformations (4) and (7) are such that they precisely generate uniform symplectic rotations of the appropriate phase-space variables for the vectors and scalars of the $N=2$ vector multiplets, so that we are in fact dealing with canonical transformations. Also for the fermions, the symplectic transformations induce corresponding canonical transformations.

In (7) we included a change of F_I. Because the matrix is symplectic, one can show that the new quantities $\tilde{F}_I$ can be written as the derivatives of a new function $\tilde{F}(\tilde{X})$. The new but equivalent set of equations of motion one obtains by means of the symplectic transformation (properly extended to other fields), follows from a Lagrangian based on $\tilde{F}$. It is possible to integrate (7) and one finds

$$\begin{aligned} \tilde{F}(\tilde{X}) &= F(X) - \tfrac{1}{2}X^I F_I(X) \\ &\quad + \tfrac{1}{2}\Big[(U^{\mathrm{T}}W)_{IJ}X^IX^J + (U^{\mathrm{T}}V + W^{\mathrm{T}}Z)_I{}^J X^I F_J + (Z^{\mathrm{T}}V)^{IJ}F_IF_J\Big] , \end{aligned} \tag{8}$$

up to a constant and terms linear in the $\tilde{X}^I$. In the coupling to supergravity, where the function must be homogeneous of second degree, such terms are obviously excluded[2].

The above expression (8) is not always so useful, as it requires substituting $\tilde{X}^I$ in terms of X^I, or vice versa. When F remains unchanged, $\tilde{F}(\tilde{X}) = F(\tilde{X})$, the

[2]The terms linear in $\tilde{X}$ in (8) are associated with constant translations in $\tilde{F}_I$ in addition to the symplectic rotation shown in (7). Likewise one may introduce constant shifts in $\tilde{X}^I$. Henceforth we ignore these shifts, which are excluded for local supersymmetry, even in the presence of a background. Constant contributions to $F(X)$ are always irrelevant.

theory is *invariant* under the corresponding transformations, but again this is hard to verify explicitly in this form. A more convenient method instead, is to verify that the substitution $X^I \to \tilde{X}^I$ into the derivatives $F_I(X)$ correctly induces the symplectic transformations on the periods (X^I, F_J). In the next section we present a few examples where the new function is determined explicitly.

Clearly $F(X)$ does not transform as a function under symplectic transformations and it is this aspect that is central to what follows. However, (8) shows immediately that the combination

$$F(X) - \tfrac{1}{2} X^I F_I(X) \tag{9}$$

does transform as a function under the symplectic transformations, i.e., as $\tilde{f}(\tilde{X}) = f(X)$. There are more quantities that transform as functions under symplectic transformation, but they are usually not holomorphic. In the coupling to supergravity (9) vanishes identically by virtue of the homogeneity of $F(X)$. It is no coincidence that in the context of the effective action of supersymmetric Yang-Mills theories, (9) is often expressible in terms of the moduli (this happens whenever the (X^I, F_J) satisfy certain Picard-Fuchs equations) and is therefore a function that must be invariant under the group of monodromy transformations, which is a subgroup of the symplectic group [6,12-14,11].

3. An example

To exhibit the effect of the symplectic reparametrizations consider the following example that decribes supergravity coupled to two vector multiplets. Hence $n = 2$ and the symplectic transformations constitute the group $Sp(6;\mathbf{R})$. The holomorphic function is taken equal to

$$F(X) = -\frac{X^1 (X^2)^2}{X^0}\,, \tag{10}$$

and gives rise to the following Kähler potential,

$$K = -\ln(S+\bar{S})(T+\bar{T})^2\,, \quad \text{where} \quad iS = X^1/X^0, \quad iT = X^2/X^0\,. \tag{11}$$

In supergravity the quantities X^I are sections of a complex line bundle, which can be parametrized in terms of the coordinates S and T. The latter are the complex scalar fields belonging to the two vector multiplets.

This example arises in the effective field theory corresponding to a compactification of the heterotic string on $K_3 \times T^2$. Apart from the graviphoton there are three additional abelian vector fields whose scalar partners are the dilaton S and the toroidal moduli T and U. However, one of the vector multiplets has been frozen such that $U = T$. This example has been used to test certain consequences of string-string duality [15-17], following a proposal for dual pairs of $N = 2$ string vacua in [18].

The corresponding Kähler manifold has $SU_{\rm S}(1,1) \times SU_{\rm T}(1,1)$ isometries. The first

$SU(1,1)$ group correponds to S-duality and is generated by the symplectic matrices

$$\begin{pmatrix} d & c & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & -\frac{1}{2}c \\ 0 & 0 & 0 & a & -b & 0 \\ 0 & 0 & 0 & -c & d & 0 \\ 0 & 0 & -2b & 0 & 0 & a \end{pmatrix},$$

where always $ad-bc=1$. This leads to the following transformations of the X^I,

$$X^0 \to dX^0 + cX^1, \quad X^1 \to bX^0 + aX^1, \quad X^2 \to \frac{X^2}{X^0}(dX^0 + cX^1)\,,$$

from which one determines the typical $SU(1,1)$ transformations

$$iS \to \frac{aiS+b}{ciS+d}\,, \qquad T \to T\,. \tag{12}$$

The second $SU(1,1)$ group corresponds to T-duality and is generated by the symplectic matrices

$$\begin{pmatrix} d^2 & 0 & 2cd & 0 & -c^2 & 0 \\ 0 & d^2 & 0 & c^2 & 0 & -cd \\ bd & 0 & ad+bc & 0 & -ac & 0 \\ 0 & b^2 & 0 & a^2 & 0 & -ab \\ -b^2 & 0 & -2ab & 0 & a^2 & 0 \\ 0 & -2bd & 0 & -2ac & 0 & ad+bc \end{pmatrix},$$

where a, b, c and d now parametrize the second $SU(1,1)$ group and are again subject to $ad-bc=1$. These T-duality transformations give rise to

$$\begin{aligned} X^0 &\to \frac{1}{X^0}(dX^0 + cX^2)^2\,, \\ X^1 &\to \frac{X^1}{(X^0)^2}(dX^0 + cX^2)^2\,, \\ X^2 &\to \frac{dX^0 + cX^2}{X^0}(bX^0 + aX^2)\,, \end{aligned} \tag{13}$$

thus leading to

$$S \to S\,, \qquad iT \to \frac{aiT+b}{ciT+d}\,. \tag{14}$$

The above transformations constitute symmetries of the theory and therefore do not cause a change of the function $F(X)$. By this, we do *not* wish to imply that $F(X)$

is invariant under the above substitutions, but rather that the new function following from the symplectic reparametrization according to (8), coincides with the previous one.

Of course, one may also consider symplectic transformations that do not have this property and therefore define a reparametrization rather than a symmetry. One such example is the symplectic rotation defined by

$$\tilde{X}^2 = \alpha F_2\,, \qquad \tilde{F}_2 = -\frac{1}{\alpha} X^2 + \beta F_2\,, \tag{15}$$

with all other fields unchanged. From this rotation we easily determine the submatrices U, V, W and Z of the symplectic matrix so that we can construct the new function according to (8). It is equal to

$$\tilde{F}(\tilde{X}) = \frac{1}{4\alpha^2}\frac{\tilde{X}^0(\tilde{X}^2)^2}{\tilde{X}^1} + \frac{\beta}{2\alpha}(\tilde{X}^2)^2\,. \tag{16}$$

Another symplectic transformation, defined by

$$\begin{aligned}
\tilde{X}^0 &= X^0\,, & \tilde{F}_0 &= F_0\,,\\
\tilde{X}^1 &= \alpha_1 F_2\,, & \tilde{F}_1 &= -\frac{1}{\alpha_1}X^2 + \frac{\beta}{\alpha_1}F_1 + \gamma F_2\,,\\
\tilde{X}^2 &= \alpha_2 F_1\,, & \tilde{F}_2 &= -\frac{1}{\alpha_2}X^1 + \delta F_1 + \frac{\beta}{\alpha_2}F_2\,.
\end{aligned} \tag{17}$$

leads to a new function that is qualitatively even more different,

$$\tilde{F}(\tilde{X}) = \pm\frac{1}{\alpha_1\alpha_2}\sqrt{-\alpha_2\tilde{X}^0(\tilde{X}^1)^2\tilde{X}^2} + \frac{\gamma}{2\alpha_1}(\tilde{X}^1)^2 + \frac{\beta}{\alpha_1\alpha_2}\tilde{X}^1\tilde{X}^2 + \frac{\delta}{2\alpha_2}(\tilde{X}^2)^2\,. \tag{18}$$

Of course, the symmetry transformations in terms of the new coordinates are different. For instance, in the case above with $\alpha_1 = \alpha_2 = -1$ and $\beta = \gamma = \delta = 0$, the symplectic matrices corresponding to S- and T-duality take the form

$$\begin{pmatrix} d & 0 & 0 & 0 & 0 & c\\ 0 & a & 0 & 0 & 2b & 0\\ 0 & 0 & d & c & 0 & 0\\ 0 & 0 & b & a & 0 & 0\\ 0 & \frac{1}{2}c & 0 & 0 & d & 0\\ b & 0 & 0 & 0 & 0 & a \end{pmatrix}\,, \qquad \begin{pmatrix} d^2 & 0 & c^2 & 0 & 2cd & 0\\ 0 & ad+bc & 0 & 2ac & 0 & 2bd\\ b^2 & 0 & a^2 & 0 & 2ab & 0\\ 0 & ab & 0 & a^2 & 0 & b^2\\ bd & 0 & ac & 0 & ad+bc & 0\\ 0 & cd & 0 & c^2 & 0 & d^2 \end{pmatrix}\,.$$

For certain symplectic rotations the transformation X^I to $\tilde{X}^I$ is singular and in this case there is no new function F, although the $(\tilde{X}^I, \tilde{F}_I)$ still exist. The latter is merely a technical problem as the full Lagrangian can still be written down consistently in terms of the $(\tilde{X}^I, \tilde{F}_I)$ and their derivatives [19]. However, in the presence of

charges one no longer has the possibility of performing arbitrary symplectic transformations. In the context of our example a symplectic rotation that does not lead to a new function F, is, for instance,

$$\hat{X}^1 = F_1 \,, \qquad \hat{F}_1 = -X^1 \,, \tag{19}$$

with the other fields unchanged. On the basis of the $(\tilde{X}^I, \tilde{F}_I)$, the S- and T-duality transformations are described by the following two symplectic matrices, respectively,

$$\begin{pmatrix} & & & 0 & -c & 0 \\ & d\mathbf{1} & & -c & 0 & 0 \\ & & & 0 & 0 & -\frac{1}{2}c \\ 0 & -b & 0 & & & \\ -b & 0 & 0 & & a\mathbf{1} & \\ 0 & 0 & -2b & & & \end{pmatrix}, \begin{pmatrix} d^2 & -c^2 & 2cd & 0 & 0 & 0 \\ -b^2 & a^2 & -2ab & 0 & 0 & 0 \\ bd & -ac & ad+bc & 0 & 0 & 0 \\ 0 & 0 & 0 & a^2 & -b^2 & -ab \\ 0 & 0 & 0 & -c^2 & d^2 & cd \\ 0 & 0 & 0 & -2ac & 2bd & ad+bc \end{pmatrix}.$$

In this basis the S-duality transformations leave the action invariant provided we impose the restriction $c = 0$. There is no symplectic basis in which S-duality is an invariance of the action. This follows from the fact that $SU_{\rm S}(1,1)$ is embedded into $Sp(6,\mathbf{R})$ according to $\mathbf{2}\oplus\mathbf{2}\oplus\mathbf{2}$. On the other hand, the T-duality transformations leave the action manifestly invariant in this basis. This is possible, because $SU_{\rm T}(1,1)$ is embedded according to $\mathbf{3}\oplus\mathbf{3}$. With these observations we conclude our discussion of the example.

4. Symplectic covariance and holomorphy

The symplectic transformations can also be performed in the presence of chiral background fields as well as in a conformal supergravity background. To couple supersymmetric vector multiplets to (scalar) chiral background fields is straightforward as one can simply incorporate additional chiral fields Φ into the function F that appears in the integrand of (1). Also the coupling to conformal supergravity is known [20]. We draw attention to the fact that the W^I are reduced, while the Φ can be either reduced or general chiral fields.

There are a number of situations where chiral backgrounds are relevant. In supersymmetric theories many of the parameters (coupling constants, masses) can be regarded as background fields that are frozen to constant values (so that supersymmetry is left intact). Because these background fields correspond to certain representations of supersymmetry, the way in which they appear in the theory – usually both perturbatively as well as nonperturbatively – is restricted by supersymmetry. In this way we may derive restrictions on the way in which parameters can appear. An example is, for instance, the coupling constant and θ-angle of a supersymmetric gauge theory, which can be regarded as a chiral field frozen to a complex constant $iS = \theta/2\pi + i4\pi/g^2$. Supersymmetry now requires that the function $F(X)$ depends

on S, but *not* on its complex conjugate. This strategy of introducing so-called *spurion* fields is not new. In the context of supersymmetry it has been used in, for instance, [21, 22, 23] to derive nonrenormalization theorems and even exact results.

Spurion fields can also be used for mass terms of hypermultiplets. When considering the effective action after integrating out the hypermultiplets, the dependence on these mass parameters can be incorporated in chiral background fields. In this case the background fields must be restricted to reduced chiral fields. In the previous example this restriction was optional. On the other hand, it may also be advantageous to not restrict the background fields to constant values, in order to study an explicit breaking of supersymmetry [24, 25].

We add that this strategy of using background (spurion) fields is very natural from the point of view of string theory, where the moduli fields, which characterize the parameters of the (supersymmetric) low-energy physics, reside in supermultiplets. In heterotic $N = 2$ compactifications the background field S introduced above coincides with the complex dilaton field, which comprises the dilaton and the axion, and belongs to a vector multiplet. The dilaton acts as the loop-counting parameter for string perturbation theory. Although the full supermultiplet that contains the dilaton is now physical, the derivation of nonrenormalization theorems can proceed in the same way [26, 8]. We should stress here that when restricting the background to a reduced chiral multiplet, one can just treat it as an additional (albeit external) vector multiplet. Under these circumstances one may consider extensions of the symplectic transformations that involve also the background itself. Of course, when freezing the background to constant values, one must restrict the symplectic transformations accordingly. The above strategy is especially useful when dealing with anomalous symmetries. By extending anomalous transformations to the background fields, the variation of these fields can compensate for the anomaly. The extended non-anomalous symmetry becomes again anomalous once the background is frozen to a contant value. This strategy can be advantageous when dealing with massive hypermultiplets.

Another context where chiral backgrounds are relevant concerns the coupling to the Weyl multiplet, which involves interactions of vector multiplets to the square of the Riemann tensor. In this case the scalar chiral background is not reduced and is proportional to the square of the Weyl multiplet. Here the strategy is not, of course, to freeze the background to a constant, but one is interested in more general couplings with conformal supergravity. In the context of string theory the coefficient functions in terms of the Weyl background were studied and evaluated from certain type-II string amplitudes [28]. An intriguing feature is that these functions can be identified with the topological partition functions of a two-dimensional twisted nonlinear sigma model defined on a Calabi-Yau target space. These partition functions are obtained by appropriately integrating over genus-g Riemann surfaces. However, they do not depend holomorphically on the Calabi-Yau moduli, but there is a holomorphic anomaly due to certain nongeneric contributions coming the boundary of

the moduli space underlying the Riemann surfaces [29]. In string theory these can be understood from the propagation of massless states [30]. Here we study similar coefficient functions, but with respect to general chiral backgrounds, and derive very similar results by insisting on a certain behaviour under symplectic transformations.

It is possible to perform the standard analysis of the symplectic reparametrizations in the presence of chiral background fields starting from functions F that depend both on the gauge superfield strengths W^I and on the background field Φ in a way that is a priori unrestricted. Then one can proceed exactly as before and examine the equivalence classes in the presence of the background. The transformation rules, however, will also depend on the background fields. This does not affect the derivation, but there are a number of new features. We assume the presence of a single chiral scalar background field (the generalization to more background fields is straightforward) whose lowest-dimensional bosonic component is denoted by A (for details, see [11]). It turns out that the symplectic reparametrizations are fully consistent in the chiral background. The function $F(X, A)$ still changes according to (8), where A remains unaffected although the transformations themselves depend on A. As before, F does not transform as a function under symplectic transformations, but it is possible to identify certain quantities that do transform in this way. One of them is, for instance, the Kähler potential, but the Kähler potential is never holomorphic. This lack of holomorphy is not an exception. There are very few quantities that transform as functions under symplectic transformations and are holomorphic at the same time. Two such functions are

$$F(X,A) - \tfrac{1}{2} X^I F_I(X,A)\,, \quad \text{and} \quad F_A(X,A)\,,$$

where F_A denotes the first derivative of F with respect to the background field A. All other symplectic functions that we have been able to identify, are not holomorphic. In particular, higher derivatives of $F(X, A)$ with respect to the background A do not transform as functions under holomorphic parametrizations. This conclusion is rather disturbing when considering symplectic transformations that constitute an invariance. In that situation we have $\tilde{F}(\tilde{X}, A) = F(\tilde{X}, A)$, but in spite of that, the coefficient functions (proportional to multiple derivatives of $F(X, A)$ with respect to the background) are not invariant *functions* under the corresponding transformations. This is only so for the first derivative F_A.

It turns out, however, that one can systematically modify the multiple-A derivatives of F, such that they will transform as functions under symplectic transformations. Naturally such modified functions are expected to arise when evaluating the coefficient functions directly from some underlying theory, such as string theory. Here we should stress that, in the context of the Wilsonian action, the original (holomorphic) coefficient functions do not directly correspond to physically relevant quantities. Therefore they do not have to be invariant under the symmetries associated with a subgroup of the symplectic transformations (such as the target-space dualities in string theory). In the remainder of this section, we will be completely general and

construct a hierarchy of modified coefficient functions transforming as functions under the symplectic group. Subsequently we derive the holomorphic anomaly equation pertaining to these functions.

The construction of the modified multiple derivatives, which define the coefficient functions when expanding order-by-order in the background, proceeds as follows. First, assume that $G(X,A)$ transforms as a function under symplectic transformations. Then one readily proves that also $\mathcal{D}G(X,A)$ transforms as a symplectic function, where

$$\mathcal{D} \equiv \frac{\partial}{\partial A} + iF_{AI}N^{IJ}\frac{\partial}{\partial X^J}\,, \tag{20}$$

and

$$N_{IJ} \equiv 2\,\mathrm{Im}\,F_{IJ}\,, \qquad N^{IJ} \equiv [N^{-1}]^{IJ}\,.$$

We note that $\mathcal{D}$ and $N^{IJ}\partial_J$ commute. Using (20) one can directly write down a hierarchy of functions which are modifications of multiple derivatives $F_{A\cdots A}$,

$$F^{(n)}(X,A) \equiv \frac{1}{n!}\mathcal{D}^{n-1}F_A(X,A)\,, \tag{21}$$

where we included an obvious normalization factor. All the $F^{(n)}$ transform as functions under symplectic functions. However, except for $F^{(1)}$, they are not holomorphic. The lack of holomorphy is governed by the following equation ($n>1$),

$$\frac{\partial F^{(n)}}{\partial \bar X^I} = \tfrac{1}{2}\bar F_I{}^{JK}\sum_{r=1}^{n-1}\frac{\partial F^{(r)}}{\partial X^J}\,\frac{\partial F^{(n-r)}}{\partial X^K}\,, \tag{22}$$

where $\bar F_I{}^{JK} = \bar F_{ILM}\,N^{LJ}N^{MK}$. This equation resembles the equation for the holomorphic anomaly found in [29] for the topological partition functions of twisted Calabi-Yau nonlinear sigma models. The latter equation exhibits two terms, however, and only one of them coincides with the right-hand side of (22). This is the term that arises from Riemann surfaces that tend to be pinched into two disconnected surfaces. The missing term corresponds to pinchings of a closed loop, which lowers the genus by one unit. In the context of our derivation the latter term is of a different nature than the first one, as the genus is tied to an expansion order-by-order in the background field, but it can probably be incorporated by making further modifications to the coefficient functions. The fact that only one term occurs in the above anomaly equation, implies that no integrability relation can be derived for $F^{(1)}$, which remains holomorphic here.

We should stress that (22) was obtained in a very general context and applies to both rigid and local $N=2$ supersymmetry. In the latter case we have to convert to holomorphic sections $X^I(z)$ and also the $F^{(n)}$ can be regarded as sections of a complex line bundle. This requires to set $A=0$ in the coefficient functions, so that we have functions that are homogeneous in the X^I. For the conversion of (22) to the case of local supersymmetry, one may conveniently make use of the formula

$$N^{IJ} = e^K\Big[g^{A\bar B}\,(\partial_A+\partial_A K)X^I(z)\,(\partial_{\bar B}+\partial_{\bar B}K)\bar X^J(\bar z) - X^I(z)\,\bar X^J(\bar z)\Big]\,, \tag{23}$$

where $K(z,\bar z)$ and $g_{A\bar B}(z,\bar z)$ are the Kähler potential and metric, respectively. However, there is more to the coefficient functions than their dependence on the coordinates z. In order to derive all the corresponding couplings one needs the full dependence on the sections $X^I(z)$, which is not encoded in the topological partition functions.

The holomorphic anomaly can thus be viewed as arising from a conflict between the requirements of holomorphy and of a proper (covariant) behaviour under symplectic transformations. The nonholomorphic modifications exhibited above can be regarded as (part of) the threshold corrections that arise due to the propagation of massless states [30]. Although the modifications presented above are not unique (we can always add to them some other symplectic function), it seems that they are in some sense universal, at least in the context of a background expansion. This would explain why, on the one hand, they can be obtained in such a general setting, while, on the other hand, they precisely reproduce one of the terms of the anomaly equation obtained in a much more specific context [29]. We should also point out that there seems a certain similarity here with the philosophy taken in [31], where the holomorphic anomaly of [29] is regarded as an obstruction to (naive) background independence, as the latter is related to the freedom of choosing a symplectic basis.

Acknowledgements
I am grateful for valuable discussions with L. Alvarez-Gaumé, R. Dijkgraaf, A. Klemm, J. Louis, D. Lüst, K. Sfetsos, S. Theisen and E. Verlinde.

References

1. S. Ferrara, J. Scherk and B. Zumino, Nucl. Phys. **B121** (1977) 393.
2. B. de Wit and A. Van Proeyen, Nucl. Phys. **B245** (1984) 89.
3. S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. **A4** (1989) 2457.
4. A. Strominger, Commun. Math. Phys. **133** (1990) 163.
5. S. Ferrara and A. Strominger, *in* Strings '89, eds. R. Arnowitt et al. (World Scientific, 1989), p. 245,
   P. Candelas and X. C. de la Ossa, Nucl. Phys. **B355** (1991) 455,
   P. Candelas, X. C. de la Ossa, P. Green and L. Parkes, Phys. Lett. **258B** (1991) 118; Nucl. Phys. **B359** (1991) 21.
6. N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19; Erratum **B430** (1994) 485; **B431** (1994) 484.
7. A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. **B344** (1995) 169.
   P. Argyres and A. Faraggi, Phys. Rev. Lett. **73** (1995) 3931.
8. B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. **B451** (1995) 53.
9. I. Antoniadis, S. Ferrara, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. **B447** (1995) 35.
10. G. Lopes Cardoso, D. Lüst and T. Mohaupt, Nucl. Phys. **B455** (1995) 131.

11. B. de Wit, $N = 2$ *Electric-magnetic duality in a chiral background*, Proc. of the 29^{th} International Symposium Ahrenshoop on the Theory of Elementary Particles, Buckow, August 29 - September 2, 1995; to appear in Nucl. Phys. **B** Proc. Suppl. (hep-th/9602060).
12. M. Matone, Phys. Lett. **B357** (1995) 342.
13. J. Sonnenschein, S. Theisen and S. Yankielowicz, Phys. Lett. **B367** (1996) 145.
14. T. Eguchi and S.-K. Yang, *Prepotentials of $N = 2$ supersymmetric gauge theories and soliton equations*, preprint UT-728, hep-th/9510183.
15. V. Kaplunovsky, J. Louis and S. Theisen, Phys. Lett. **357B** (1995) 71.
16. A. Klemm, W. Lerche and P. Mayr, Phys. Lett. **357B** (1995) 313.
17. I. Antoniadis, E.Gava, K. Narain and T.R. Taylor, Nucl. Phys. **B455** (1995) 109.
18. S. Kachru and C. Vafa, Nucl. Phys. **B450** (1995) 69.
19. A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, Nucl. Phys. **B444** (1995) 92.
20. B. de Wit, P.G. Lauwers and A. Van Proeyen, Nucl. Phys. **B255** (1985) 569.
21. M.A. Shifman and A.I. Vainshtein, Nucl. Phys. **B277** (1986) 456; **B359** (1991) 571.
22. D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, Phys. Rep. **162** (1988) 169.
23. K. Intrilligator, R.G. Leigh and N. Seiberg, Phys. Rev. **D50** (1994) 1092.
24. L. Girardello and M.T. Grisaru, Nucl. Phys. **B194** (1982) 65.
25. L. Alvarez-Gaumé, J. Distler, C. Kounnas and M. Mariño, *Softly broken $N = 2$ QCD*, CERN-TH/9669, to appear.
26. H.P. Nilles, Phys. Lett. **180B** (1986) 240.
27. E.A. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. **B182** (1981) 173.
28. I. Antoniadis, E.Gava, K. Narain and T.R. Taylor, Nucl. Phys. **B413** (1994) 162.
29. M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Nucl. Phys. **B405** (1993) 279; Commun. Math. Phys. **165** (1994) 311.
30. L.J. Dixon, V.S. Kaplunovsky and J. Louis, Nucl. Phys. **B355** (1991) 649.
31. E. Witten, *Quantum background independence in string theory*, hep-th/9306122.